Pearson Edexcel Level 3

GCE Mathematics

Advanced

Paper 2: Pure Mathematics

PMT Mock 3

Time: 2 hours	**Paper Reference(s)** **9MA0/02**

You must have:
Mathematical Formulae and Statistical Tables, calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
- There are 14 questions in this paper. The total mark is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

1. Given that a is a positive constant,

a. Sketch the graph with equation

$$y = |a - 2x|$$

Show on your sketch the coordinates of each point at which the graph crosses the x-axis and y-axis.

(2)

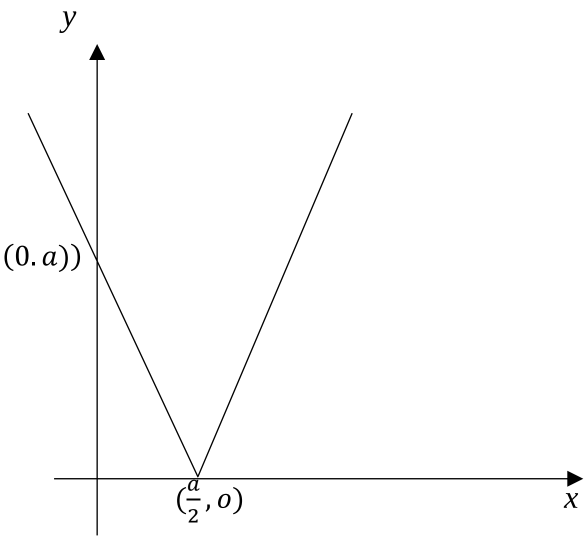
b. Solve the inequality $|a - 2x| > x + 2a$

(3)

a. B1 V shape with vertex on x-axis but not at the origin

B1 Correct V shape with $(0, a)$ or just a and $(\frac{a}{2}, 0)$ or just $\frac{a}{2}$ marked in the correct places.

Left branch must cross or touch the y-axis.



b. M1 Attempts to solve $a - 2x = x + 2a \Rightarrow x = \cdots$ or $-a + 2x = x + 2a \Rightarrow x = \cdots$

e.g. $a - 2x = x + 2a \Rightarrow -3x = a \Rightarrow x = -\frac{a}{3}$

or $-a + 2x = x + 2a \Rightarrow 2x - x = 2a + a \Rightarrow x = 3a$

M1 Attempts to solve $a - 2x = x + 2a \Rightarrow x = \cdots$ and $-a + 2x = x + 2a \Rightarrow x = \cdots$

A1 Chooses outside region giving correct answer only. $x < -\frac{a}{3}$ or $x > 3a$

Allow alternative e.g. $x < -\frac{a}{3} \cup x > 3a$

(Total for Question 1 is 5 marks)

2. Solve

$$4^{x-3} = 6$$

giving your answer in the form $a + b\log_2 3$, where a and b are constants to be found.

(4)

M1 Uses logs in an attempt to solve the equation.
e.g. takes log base 4 and obtains $(x-3) = \log_4 6$
Alternative takes logs (any base) to obtain $(x-3)\log 4 = \log 6$ and proceeds to $x - 3 = \frac{\log 6}{\log 4}$

A1 Changes log bases to 2 and obtains $(x-3) = \frac{\log_2 6}{\log_2 4}$

M1 This mark would be awarded for $\frac{\log_2 6}{\log_2 4} = \frac{\log_2(2\times 3)}{\log_2 2^2} = \frac{\log_2 2+\log_2 3}{2\log_2 2} = \frac{1+\log_2 3}{2}$

A1 Correct answer only that leads to a value of x.

e.g. $x - 3 = \frac{1+\log_2 3}{2} \quad \Rightarrow \quad x = \frac{7}{2} + \frac{1}{2}\log_2 3$

(Total for Question 2 is 4 marks)

3. Given that

$$y = \frac{1}{3}x^3$$

use differentiation from first principle to show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2$$

(3)

M1 Begins the process by writing down the gradient of the chord and attempts to expand the correct bracket.

e.g. $\frac{\frac{1}{3}(x+h)^3 - \frac{1}{3}x^3}{h} = \frac{\frac{1}{3}(x^3+3x^2h+3xh^2+h^3) - \frac{1}{3}x^3}{h} =$

A1 Reaches a correct fraction or equivalent with the x^3 terms cancelled out

e.g. $\frac{\frac{1}{3}(x^3+3x^2h+3xh^2+h^3) - \frac{1}{3}x^3}{h} = \frac{\cancel{\frac{1}{3}x^3}+x^2h+xh^2+\frac{1}{3}h^3-\cancel{\frac{1}{3}x^3}}{h} = x^2 + xh + \frac{1}{3}h^2$

A1 Completes the process by applying a limiting argument and deduces that $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2$ with no errors seen.

The "$\frac{\mathrm{d}y}{\mathrm{d}x} =$" doesn't have to appear but there must be something equivalent e.g. "$\mathrm{f}'(x) =$" or "Gradient =" which can appear anywhere in their working. If $\mathrm{f}'(x)$ is used then there is no requirement to see f(x) defined first. Condone e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} \to x^2$ or $\mathrm{f}'(x) \to x^2$

Condone missing brackets so allow e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{x^2h+xh^2+\frac{1}{3}h^3}{h} = \lim_{h\to 0} x^2 + xh + \frac{1}{3}h^2 = x^2$

Do not allow $h = 0$ if there is never a reference to $h \to 0$

e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{x^2h+xh^2+\frac{1}{3}h^3}{h} = \lim_{h\to 0} x^2 + x(0) + \frac{1}{3}(0)^2 = x^2$ is acceptable

but e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2h+xh^2+\frac{1}{3}h^3}{h} = x^2 + x(0) + \frac{1}{3}(0)^2 = x^2$ is not if there is no $h \to 0$ seen.

The $h \to 0$ does not need to be present throughout the proof. e.g. on every line.

They must reach $x^2 + xh + \frac{1}{3}h^2$ at the end and not $\frac{x^2h+xh^2+\frac{1}{3}h^3}{h}$ to complete the limiting argument.

(Total for Question 3 is 3 marks)

4. A sequence a_1, a_2, a_3 is defined by

$$a_n = \sin^2\left(\frac{n\pi}{3}\right)$$

Find the exact values of

a. i) a_1
ii) a_2
iii) a_3

(3)

b. Hence find the exact value of

$$\sum_{n=1}^{100}\left\{n + \sin^2\left(\frac{n\pi}{3}\right)\right\}$$

(3)

a. i. B1 $a_1 = \frac{3}{4}$ e.g. $a_1 = \sin^2\left(\frac{\pi}{3}\right) = \frac{3}{4}$

ii. B1 $a_2 = \frac{3}{4}$ e.g. $a_2 = \sin^2\left(\frac{2\pi}{3}\right) = \frac{3}{4}$

iii. B1 $a_3 = 0$ e.g. $a_3 = \sin^2\left(\frac{3\pi}{3}\right) = 0$

b. M1 Attempts a correct method to find the sum of $1 + 2 + 3 + \cdots + 100$

e.g. $\frac{100}{2}(2 + 99)$ or $\frac{100}{2}(1 + 100)$

M1 Attempts a correct method to find $\sum_{n=1}^{100} \sin^2\left(\frac{n\pi}{3}\right)$

e.g. $66 \times \frac{3}{4} + \frac{3}{4}$ or $33 \times \frac{3}{4} + 33 \times \frac{3}{4} + \frac{3}{4}$

Must be a correct method for the correct sequence.

e.g. $\frac{3}{4} + \frac{3}{4} + 0 + \frac{3}{4} + \frac{3}{4} + 0 + \frac{3}{4} + \frac{3}{4} + 0 + \cdots$

A1 $5050 + \frac{201}{4} = \frac{20401}{4}$ or exact equivalent e.g. 5100.25

(Total for Question 4 is 6 marks)

5. The table below shows corresponding values of x and y for $y = \log_3(x)$
The values of y are given to 2 decimal places as appropriate.

x	3	4.5	6	7.5	9
y	1	1.37	1.63	1.83	2

a. Obtain an estimate for $\int_3^9 \log_3(x)\ \ \mathrm{d}x$, giving your answer to two decimal places.

(3)

Use your answer to part (a) and making your method clear, estimate

b. i) $\int_3^9 \log_3\sqrt{x}\ \ \mathrm{d}x$

ii) $\int_3^{18} \log_3(9x^3)\ \ \mathrm{d}x$

(3)

a. B1 States and uses $h = 1.5$

M1 A full attempt at the trapezium rule.

Look for $\frac{their\ h}{2}\{1 + 2 + 2 \times (1.37 + 1.63 + 1.83)\}$ but condone copying slips

Allow this if they add the areas of individual trapezia. e.g.

$$\frac{their\ h}{2}\{1 + 1.37\} + \frac{their\ h}{2}\{1.37 + 1.63\} + \frac{their\ h}{2}\{1.63 + 1.83\} + \frac{their\ h}{2}\{1.83 + 2\}$$

A1 9.50

b. i. B1 awrt 4.75 e.g. $\frac{1}{2} \times \log_3 x = \frac{1}{2} \times 9.50 = 4.75$

ii. M1 States and implies that $\log_3 9x^3 = 2 + 3\log_3 x$ and

$\int_3^{18} \log_3(9x^3)\ \ \mathrm{d}x = [2x]_3^{18} + 3 \times 9.50 = \cdots$ or equivalent work for

finding the area of the rectangle as 2×15

A1 Correct working followed by awrt 58.5

e.g. $[2x]_3^{18} + 3 \times 9.50 = 2 \times 18 - 2 \times 3 + 3 \times 9.50 = 58.5$

(Total for Question 5 is 6 marks)

6.

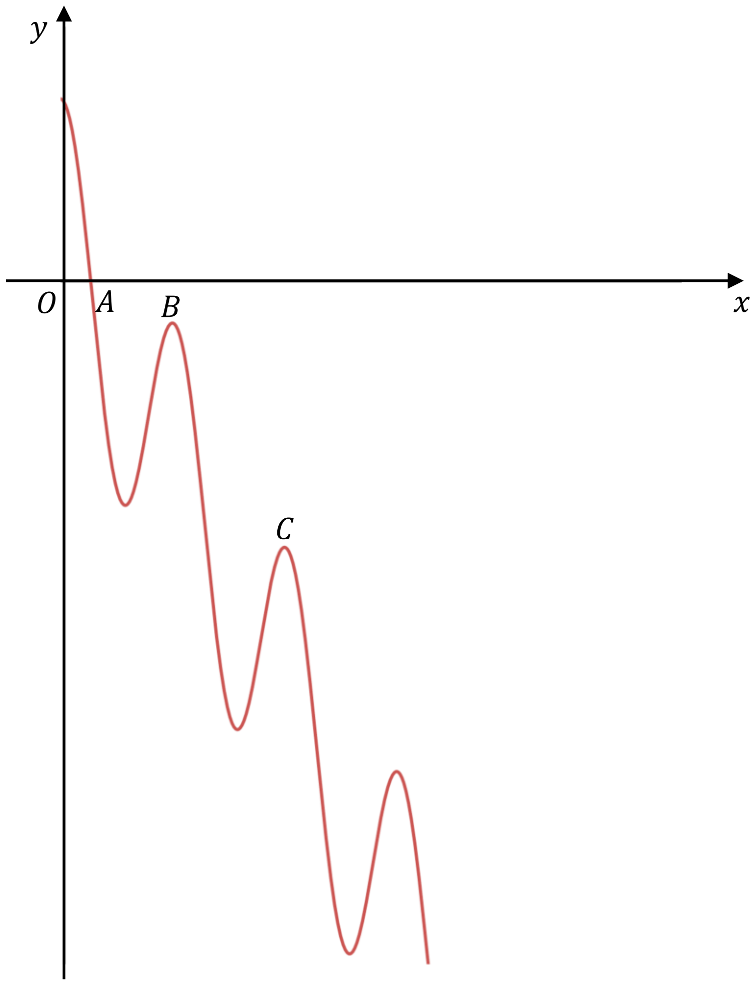


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$f(x) = 4\cos 2x - 2x + 1 \qquad x > 0$$

and where x is measured in radians.

The curve crosses the x-axis at the point A, as shown in figure 1.

Given that x-coordinate of A is α

a. show that α lies between 0.7 and 0.8

(2)

Given that x-coordinates of B and C are β and γ respectively and they are two smallest values of x at which local maxima occur

b. find, using calculus, the value of β and the value of γ, giving your answers to 3 significant figures.

(5)

c. taking $x_0 = 0.7$ or 0.8 as a first approximation to α, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation to α. Show, your method and give your answer to 2 significant figures.

(2)

a. M1 Attempts the value of y at 0.7 and 0.8 with at least one correct to 1 significant figure rounded or truncated.

e.g. $y = 4\cos 2(0.7) - 2(0.7) + 1 = 0.28$ and $y = 4\cos 2(0.8) - 2(0.8) + 1 = -0.72$

A1 Both values correct to 1 significant figure rounded or truncated with reason (sign change and continuous function) and minimal conclusion (root).

e.g. $y|_{0.7} = 0.3 > 0,\ y|_{0.8} = -0.7 < 0$ and function is continuous so there is a root.

b. M1 Differentiates to obtain $A\sin 2x + B$

A1 Correct derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = -8\sin 2x - 2$.

Allow unsimplified e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = -4 \times 2 \times \sin 2x - 2$

There is no need for derivative $\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots$ just look for the expression

dM1 For the complete strategy of proceeding to a value for x.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \sin 2x = a,\ |a| < 1 \Rightarrow x = \cdots$

Look for correct order of operations, invsinα then $\div$ 2 leading to a value of x

When $\sin 2x = -\frac{1}{4}$ it is implied, for example, by awrt -0.253, awrt 3.394, awrt 6.031, awrt 9.677 or awrt 12.314

For $\sin 2x = \frac{1}{4}$ it is implied, for example, by awrt 0.253 or awrt 2.889

The calculations must be using radians. If degrees are used initially they must be converted to radians.

e.g. $2x = 6.031 \ldots \Rightarrow x = 3.02$ or $2x = 12.314 \ldots \Rightarrow x = 6.16$

A1 For recognising that either 3.02 or 6.16 is a solution to $\sin 2x = -\frac{1}{4}$

A1 States that β =3.02 and $\gamma = 6.16$ labels must be correct

c. M1 Attempts $x_1 = 0.7 - \frac{f(0.7)}{f'(0.7)}$ to obtain a value following through on their $f'(x)$ as long as it is a "changed" function.
Must be correct N-R formula used-may need to check their values.
Allow if attempted in degrees. For refence in degrees f(0.7) = 3.5998… and $f'(0.7) = -2.1954 \ldots$ and gives $x_1 = 2.3392 \ldots$
There must be clear evidence that $0.7 - \frac{f(0.7)}{f'(0.7)}$ is being attempted.
So e.g. $x_1 = 0.7 - \frac{f(0.7)}{f'(0.7)} = 0.7 - \frac{0.2798685716}{-9.88359784} = 0.728316467 = 0.728$

OR

M1 Attempts $x_1 = 0.8 - \frac{f(0.8)}{f'(0.8)}$ to obtain a value following through on their $f'(x)$ as long as it is a "changed" function.
Must be correct N-R formula used-may need to check their values.
Allow if attempted in degrees. For refence in degrees f(0.8) = 3.3984… and $f'(0.8) = -2.2233 \ldots$ and gives $x_1 = 2.3285\ldots$
There must be clear evidence that $0.8 - \frac{f(0.8)}{f'(0.8)}$ is being attempted.
So e.g. $x_1 = 0.8 - \frac{f(0.8)}{f'(0.8)} = 0.8 - \frac{-0.7167980892}{-9.996588824} = 0.7282957315 = 0.728$

A1 $x_1 =$ awrt 0.73

(Total for Question 6 is 9 marks)

7. a. Use the binomial theorem to expand

$$(8 - 3x)^{\frac{2}{3}}$$

in ascending powers of x, up to and including the term x^3, as a fully simplifying each term.

(4)

Edward, a student decides to use the expansion with $x = \frac{1}{3}$ to find an approximation for $(7)^{\frac{2}{3}}$. Using the answer to part (a) and without doing any calculations,

b. explain clearly whether Edward's approximation will be an overestimate, or, an underestimate.

(1)

a. B1 Takes out a factor of 8 and writes $(8-3x)^{\frac{2}{3}} = 4(1 \pm \cdots)^{\frac{2}{3}}$ or $8^{\frac{2}{3}}(1 \pm \cdots)^{\frac{2}{3}}$ or $2^2(1 \pm \cdots)^{\frac{2}{3}}$

M1 For an attempt at the binomial expansion of $(1+ax)^{\frac{2}{3}}$, $a \neq 1$ to form term 3 or term 4 with the correct structure. Look for the correct binomial coefficient multiplied by the corresponding power of x.

e.g. $\frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)}{2!}(\ldots x)^2$ or $\frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}(\ldots x)^3$ where $\ldots \neq 1$

A1 Correct expression for the expansion of $\left(1 - \frac{3}{8}x\right)^{\frac{2}{3}}$

e.g. $1 + \frac{2}{3} \times \left(-\frac{3}{8}x\right) + \frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)}{2!} \times \left(\pm\frac{3}{8}x\right)^2 + \frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(-\frac{3}{8}x\right)^3$

which may left unsimplified as shown but the bracketing must be correct unless any missing brackets are implied by subsequent work. If the 4 outside this expansion is only partially applied to this expansion then scores A0 but if it is applied to all terms this A1 can be implied.

OR at least 2 correct simplified terms for the final expansion from, $-x,\ -\frac{1}{16}x^2, -\frac{1}{96}x^3$

A1 $(8-3x)^{\frac{2}{3}} = 4 - x - \frac{1}{16}x^2 - \frac{1}{96}x^3$ or equivalent

OR Direct expansion in (a) can be marked in a similar way.

$$(8-3x)^{\frac{2}{3}} = (8)^{\frac{2}{3}} + \frac{2}{3} \times (8)^{\frac{-1}{3}} \times (-3x) + \left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right) \times (8)^{\frac{-4}{3}} \times \frac{(-3x)^2}{2!}$$
$$+ \left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right) \times (8)^{\frac{-7}{3}} \times \frac{(-3x)^3}{3!}$$

b. B1 States that the approximation will be an overestimate due to the fact that all terms (after the first one) in the expansion are negative or equivalent statements e.g.

- Overestimate because the terms are negative
- Overestimate as the terms are being taken away from 4

(Total for Question 7 is 5 marks)

8.

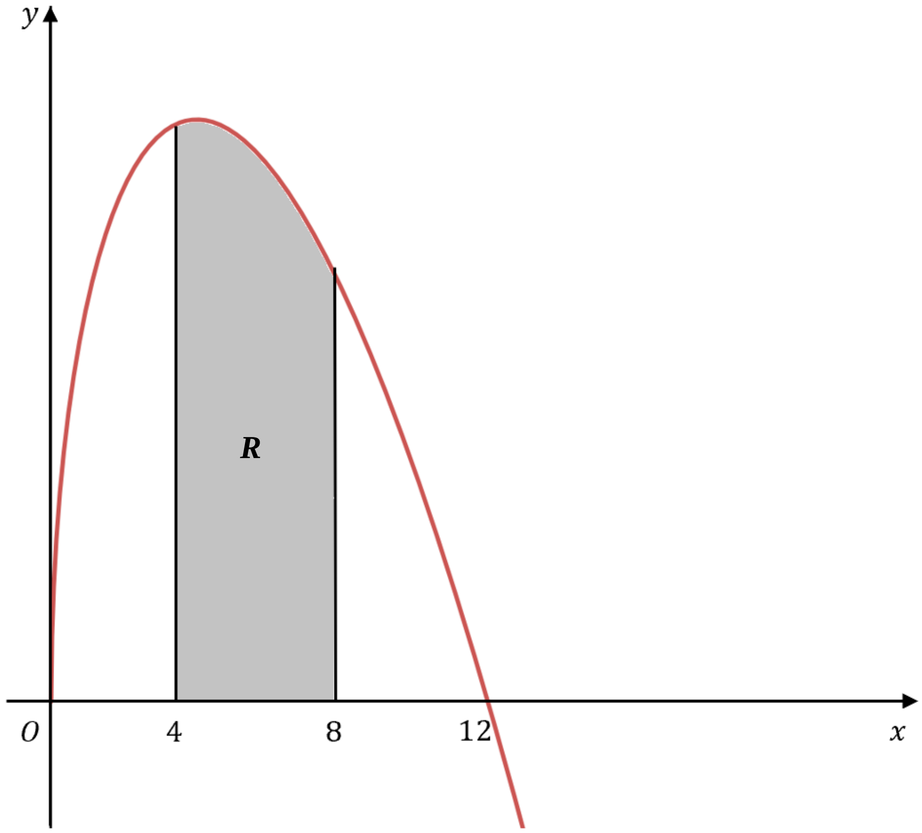


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = \frac{12x - x^2}{\sqrt{x}}, \quad x > 0$$

The region R, shows shaded in figure 2, is bounded by the curve, the line with equation $x = 4$, the x-axis and the line with equation $x = 8$.

Show that the area of the shaded region R is $\frac{128}{5}(3\sqrt{2} - 2)$.

(5)

M1 Correct attempt to write $\frac{12x-x^2}{\sqrt{x}}$ as a sum of terms with indices.

Look for the terms with the correct index. e.g. $\frac{12x}{x^{\frac{1}{2}}} - \frac{x^2}{x^{\frac{1}{2}}} = ax^{\frac{1}{2}} - bx^{\frac{3}{2}}$

M1 Integrates $x^n \to x^{n+1}$ at least 1 correct index

i.e. at least 1 of $ax^{\frac{1}{2}} \to x^{\frac{3}{2}}$, $bx^{\frac{3}{2}} \to x^{\frac{5}{2}}$

A1 $8x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} (+c)$ Allow unsimplified e.g. $\int 12x^{\frac{1}{2}} - x^{\frac{3}{2}}\ \mathrm{d}x = 12 \times \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}$

M1 Substitutes the limits 8 and 4 to their $8x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}$ and subtracts either way round.

There is no requirement to evaluate but 8 and 4 must be substituted either way round with evidence of subtraction condoning omission of brackets.

e.g. $8x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} = 8 \times 8^{\frac{3}{2}} - \frac{2}{5} \times 8^{\frac{5}{2}} - 8 \times 4^{\frac{3}{2}} - \frac{2}{5} \times 4^{\frac{5}{2}}$

A1 Correct working shown leading to $\frac{128}{5}(3\sqrt{2} - 2)$.

(Total for Question 8 is 5 marks)

9. $$f(\theta) = 4\cos\theta + 5\sin\theta \qquad \theta \in R$$

a.Express $f(\theta)$ in the form $R\cos(\theta - \alpha)$ where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

Given that

$$g(\theta) = \frac{135}{4 + f(\theta)^2} \qquad \theta \in R$$

b.find the range of g.

(2)

a. B1 $R = \sqrt{41}$ Do not allow decimals for this mark.
e.g. $\sqrt{4^2 + 5^2} = \sqrt{41}$
M1 $\tan\alpha = \pm\frac{5}{4}, \tan\alpha = \pm\frac{4}{5} \quad \Rightarrow \quad \alpha = \cdots$
If R is used to find α accept $\sin\alpha = \pm\frac{5}{R}$ or $\cos\alpha = \pm\frac{4}{R} \Rightarrow \alpha = \cdots$
e.g. $4\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha) \Rightarrow R\cos\theta\cos\alpha - \sin\theta\sin\alpha$
$\Rightarrow R\cos\alpha = 4 \Rightarrow \cos\alpha = \pm\frac{4}{R}$, $R\sin\alpha = 5 \Rightarrow \sin\alpha = \pm\frac{5}{R}$
A1 $\alpha =$ awrt 0.896

b. M1 Score for either end achieved by a correct method
Look for $\frac{135}{4}$ (implied by 33.75) , $\frac{135}{4 + their\ (\sqrt{41})^2}$, g…33.75 or g….3
A1 Accept equivalent ways of writing the interval such as [3,33.75]
Condone $3 \leq g(x) \leq 33.75$ or $3 \leq y \leq 33.75$

(Total for Question 9 is 5 marks)

10. The functions f and g are defined with their respective domains by

$$f(x) = 4 - x^2 \qquad x \in R \qquad x \geq 0$$

$$g(x) = \frac{2}{x+1} \qquad x \in R \qquad x \geq 0$$

a. Write down the range of f.

(1)

b. Find the value of fg(3)

(2)

c. Find $g^{-1}(x)$

(3)

a. B1 Correct range. Look for $f(x) \leq 4$.

Allow equivalent notation e.g. $y \leq 4, f \leq 4, y \in (-\infty, 4]$

b. M1 Full attempt at method to find fg(3) condoning slips. Implied by a correct answer or 3.75

e.g. For a correct order of operations so require an attempt to apply g (3) first and then f to their g (3)

e.g $4 - \left(\frac{2}{x+1}\right)^2 = 4 - \left(\frac{2}{3+1}\right)^2 = 3\frac{3}{4}$ or $g(3) = \frac{2}{3+1} = 0.5 \Rightarrow f(0.5) = 4 - 0.5^2 = 3.75$

A1 Correct exact value. 3.75 or $3\frac{3}{4}$ or $\frac{15}{4}$

c. M1 Changes the subject of $y = \frac{2}{x+1}$ and obtains $x = \frac{2 \pm y}{y}$ or $x = \frac{2}{y} \pm 1$ or equivalent

Alternatively changes the subject of $x = \frac{2}{y+1}$ and obtains $y = \frac{2 \pm x}{x}$ or $y = \frac{2}{x} \pm 1$ or equivalent.

A1 $g^{-1}(x) = \frac{2-x}{x}$, $g^{-1}(x) = \frac{2}{x} - 1$

Condone $y = \frac{2}{x} - 1$

B1 Correct domain. $0 < x \leq 2$ Allow equivalent notation e.g. $x \in (0,2]$

(Total for Question 10 is 6 marks)

11. Prove, using algebra that

$$n^2 + 1$$

is not divisible by 4.

(4)

M1 For the key step attempting to find $n^2 + 1$ when $n = 2k$ or $n = 2k \pm 1$

A1 Achieves $4k^2 + 1$ for $n = 2k$ or $4(k^2 + k) + 2$ for $n = 2k + 1$ or $4(k^2 - k) + 2$ for $n = 2k - 1$

dM1 Attempts find $n^2 + 1$ when $n = 2k$ and $n = 2k \pm 1$

A1 Correct work and states "is not divisible by 4" with a final conclusion e.g. so true for all $n(\in \mathbb{N})$. There should be no errors in the algebra but allow e.g. invisible brackets if they are recovered.

Alternative method:

M1 Sets up an algebraic statement in terms of a variable (integer) k or any other variable aside n that engages with divisibility by 4 in some way and can lead to a contradiction. No need for explicit statement of assumption-accept if just a suitable equation is set up. In this case supposing divisibility by 4 by stating $n^2 + 1 = 4k$

A1 Reaches $n^2 = 4k - 1 \Rightarrow n^2 = 2(2k - 1) + 1$

e.g. $n^2 = 4k - 1 = 4k - 2 + 1 = 2(2k - 1) + 1$

dM1 For a complete argument that leads to a contradiction.

Accept explanations such as "as n^2 is even then n is even hence n^2 is a multiple of 4 so $n^2 + 1$ cannot be a multiple of 4"

e.g. as n^2 is even so $n = 2m$ is even hence n^2 is a multiple of 4

As n^2 is a multiple of 4 then $n^2 + 1 = 4m^2 + 1 = 4m^2 + 2 - 1 = 2(2m^2 + 1) - 1$ cannot be a multiple of 4 so there is a contradiction.

A1 Draws the contradiction to their initial assumption and concludes the statement is true for all n. There must have been a clear assumption at the start that is contradicted, and all working must have been correct.

e.g. So the original assumption has been shown false.

Hence "$n^2 + 1$ is never divisible by 4" is true for all n

(Total for Question 11 is 4 marks)

12. A curve has equation $y = \frac{2xe^x}{x+k}$ where k is a positive constant.

i. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x(2x^2+2kx+2k)}{(x+k)^2}$

(3)

ii. Given that the curve has exactly one stationary point find the value of k.

(3)

a. i. M1 Attempts the quotient rule to obtain an expression of the form

$$\frac{(x+k)(pe^x + 2xe^x) - (2xe^x)q}{(x+k)^2}\ ,\quad p, q > 0$$

condoning bracketing errors. If the quotient rule formula is stated it must be correct.

May also use product rule to $2xe^x(x+k)^{-1}$ to obtain an expression of the form $pxe^x(x+k)^{-2} + q(x+k)^{-1}e^x(x+1)\ ,\quad p < 0, q > 0$ condoning bracketing errors. If the quotient rule formula is stated it must be correct

A1 Correct differentiation in any form with correct bracketing which may be implied by subsequent work.

e.g. $\frac{(x+k)(2e^x+2xe^x)-(2xe^x)(1)}{(x+k)^2}$ or $-2xe^x(x+k)^{-2} + 2(x+k)^{-1}e^x(x+1)$

A1 Obtains $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x(2x^2+2kx+2k)}{(x+k)^2}$

ii. B1 States that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and then $2x^2 + 2kx + 2k = 0$

M1 Attempts $b^2 - 4ac \ldots 0$ with $a = 2, b = 2k, c = 2k$ where … is either " =", <, >

e.g. $(2k)^2 - 4 \times 2 \times (2k) \ldots 0 \Rightarrow 4k^2 - 16k \ldots 0$

Alternatively attempts to complete the square and sets rhs … 0

e.g. $2[x^2 + kx + k] = 0 \Rightarrow \left(x + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2 + k = 0 \Rightarrow \left(x + \frac{k}{2}\right)^2 = \frac{k^2}{4} - k$

leading to $\frac{k^2}{4} - k = 0$

A1 $k = 4$

e.g. $4k^2 - 16k = 0 \Rightarrow 4k(k-4) = 0 \Rightarrow k = 4$

OR $\frac{k^2}{4} - k = 0 \Rightarrow k^2 - 4k = 0 \Rightarrow k(k-4) = 0 \Rightarrow k = 4$

(Total for Question 12 is 6 marks)

13. Relative to a fixed origin O

- the point P has position vector $(0, -1, 2)$
- the point Q has position vector $(1, 1, 5)$
- the point R has position vector $(3, 5, m)$

where m is a constant.

Given that P, Q and R lie on a straight line,

a. find the value of m

(3)

The line segment OQ is extended to a point T so that $\overrightarrow{RT}$ is parallel to $\overrightarrow{OP}$

b. Show that $|\overrightarrow{OT}| = 9\sqrt{3}$.

(3)

a.M1 Attempts two of the three relevant vectors by subtracting either way round.

$\pm\overrightarrow{PQ} = \pm(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

or $\pm\overrightarrow{PR} = \pm(3\mathbf{i} + 6\mathbf{j} + (m-2)\mathbf{k})$

or $\pm\overrightarrow{QR} = \pm(2\mathbf{i} + 4\mathbf{j} + (m-5)\mathbf{k})$

dM1 For the key step in using the fact that if the vectors are parallel, they will be multiples of each other to find m

e.g. $\overrightarrow{QR} = 2\overrightarrow{PQ}$ so $m - 5 = 2 \times 3$

$\overrightarrow{PR} = 3\overrightarrow{PQ}$ so $m - 2 = 3 \times 3$

$\overrightarrow{QR} = \frac{2}{3}\overrightarrow{PR}$ so $m - 5 = \frac{2}{3}(m-2)$

A1 $m = 11$

b.**Vector approach**

M1 Deduces that $\overrightarrow{OT} = \lambda\overrightarrow{OQ} = \lambda\mathbf{i} + \lambda\mathbf{j} + 5\lambda\mathbf{k}$ and

attempts $\overrightarrow{RT} = (\lambda - 3)\mathbf{i} + (\lambda - 5)\mathbf{J} + (5\lambda - m)$

dM1 Correct attempt at finding λ using the fact that $\overrightarrow{RT}$ is parallel to $\overrightarrow{OP}$

e.g. $\overrightarrow{RT} = \alpha\overrightarrow{OP} \Rightarrow (\lambda - 3)\mathbf{i} + (\lambda - 5)\mathbf{j} + (5\lambda - m)\mathbf{k} = -\alpha\mathbf{j} + 2\alpha\mathbf{k}$

$\Rightarrow (\lambda - 3) = 0$ and $(5\lambda - 11) = 2\alpha \Rightarrow \lambda = 3 \Rightarrow 4 = 2\alpha \Rightarrow \alpha = 2$

A1 $|\overrightarrow{OT}| = 9\sqrt{3}$

e.g. $\overrightarrow{OT} = 3\mathbf{i} + 3\mathbf{j} + 15\mathbf{k} \Rightarrow |\overrightarrow{OT}| = \sqrt{3^2 + 3^2 + 15^2} = 9\sqrt{3}$

Alternative:

M1 Deduces that $\overrightarrow{OT} = \lambda\overrightarrow{OQ} = \lambda\mathbf{i} + \lambda\mathbf{j} + 5\lambda\mathbf{k}$ and

attempts $\overrightarrow{OT} = \overrightarrow{OR} + \mu\overrightarrow{OP} = 3\mathbf{i} + 5\mathbf{j} + m\mathbf{k} + \mu(-\mathbf{j} + 2\mathbf{k})$

dM1 Correct attempt at finding λ or μ using the fact that $\lambda\overrightarrow{OQ} = \overrightarrow{OR} + \mu\overrightarrow{OP}$

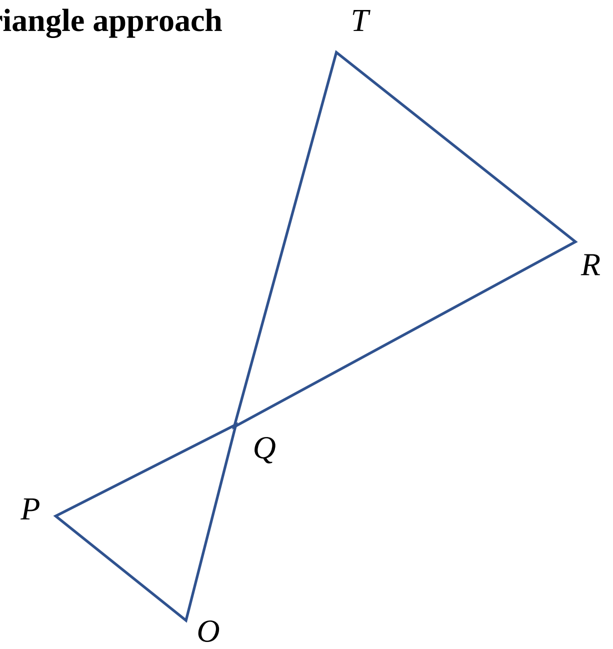
e.g. $\lambda\mathbf{i} + \lambda\mathbf{j} + 5\lambda\mathbf{k} = 3\mathbf{i} + (5-\mu)\mathbf{j} + (m+2\mu)\mathbf{k} \;\Rightarrow\; \lambda = 3$

or $\lambda = 5 - \mu$ and $5\lambda = 11 + 2\mu \;\Rightarrow\; \lambda = 3$ and $\mu = 2$

A1 $\left|\overrightarrow{OT}\right| = 9\sqrt{3}$

e.g. $\overrightarrow{OT} = 3\mathbf{i} + 3\mathbf{j} + 15\mathbf{k} \;\Rightarrow\; \left|\overrightarrow{OT}\right| = \sqrt{3^2 + 3^2 + 15^2} = 9\sqrt{3}$

Similar triangle approach



M1 For the key step in recognising that triangle QPO and triangle QRT are similar

with a ratio of lengths of 2 : 1

dM1 States or uses the fact that $\left|\overrightarrow{OT}\right| = 3\left|\overrightarrow{OQ}\right|$

A1 $\left|\overrightarrow{OT}\right| = 9\sqrt{3}$

(Total for Question 13 is 6 marks)

14. a. Express $\frac{1}{(3-x)(1-x)}$ in partial fractions.

(2)

A scientist is studying the mass of a substance in a laboratory.
The mass, *x* grams, of a substance at time *t* seconds after a chemical reaction starts is modelled by the differential equation

$$2\frac{dx}{dt} = (3-x)(1-x) \quad t \geq 0,\ 0 \leq x < 1$$

Given that when $t = 0, x = 0$

b. solve the differential equation and show that the solution can be written as

$$x = \frac{3(e^t - 1)}{3e^t - 1}$$

(5)

c. Find the mass, *x* grams, which has formed 2 seconds after the start of the reaction. Give your answer correct to 3 significant figures.

(1)

d. Find the limiting value of *x* as *t* increases.

(1)

a.M1 Correct method of partial fractions and finds at least one of *A* or *B*

e.g. $\frac{1}{(3-x)(1-x)} = \frac{A}{(3-x)} + \frac{B}{(1-x)} \quad \Rightarrow \quad 1 \equiv A(1-x) + B(3-x)$

$\Rightarrow x = 1\ , \quad 1 = 2B \quad \Rightarrow \quad A = \cdots \quad x = 3\ \ , 1 = -2A \ \Rightarrow \ B = \cdots$

A1 Correct partial fractions not just values for *A* and *B*. $\frac{\frac{1}{2}}{(3-x)} - \frac{\frac{1}{2}}{(1-x)}$

b.B1 Separates variables correctly. d*x* and d*t* should be in the correct positions, though this mark can be implied by later working.

e.g. $\int \frac{1}{(3-x)(1-x)}\,dx = \int \frac{1}{2}\,dt$ or $\int \frac{2}{(3-x)(1-x)}\,dx = \int\,dt$

M1 Correct attempt at integration of the partial fractions.

e.g. Look for $\pm\alpha\ln(3-x) \pm \beta\ln(1-x)$ where $\alpha \neq 0,\ \beta \neq 0$

A1ft Fully correct equation following through their *A* and *B* only.

e.g. $\frac{1}{2}\ln(3-x) - \frac{1}{2}\ln(1-x) = \frac{1}{2}t + c$ or $\ln(3-x) - \ln(1-x) = t + c$

M1 Attempts to find "*c*" using $t = 0, x = 0$ following an attempt at integration.

e.g. $\frac{1}{2}\ln(3) - \frac{1}{2}\ln(1) = c \quad \Rightarrow \frac{1}{2}\ln\left(\frac{3}{1}\right) = c \quad \Rightarrow \frac{1}{2}\ln 3 = c$

or $\ln(3) - \ln(1) = c \quad \Rightarrow \quad \ln\left(\frac{3}{1}\right) = c \quad \ln 3 = c$

A1 Correct processing leading to the given answer.

$$x = \frac{3(e^t - 1)}{3e^t - 1}$$

e.g. $\frac{1}{2}\ln(3-x) - \frac{1}{2}\ln(1-x) = \frac{1}{2}t + \frac{1}{2}\ln 3 \quad \Rightarrow \ln(3-x) - \ln(1-x) = t + \ln 3$

$\Rightarrow \ln(3-x) - \ln(1-x) - \ln 3 = t \quad \Rightarrow \ln\left(\frac{3-x}{3(1-x)}\right) = t \quad \Rightarrow \quad \frac{3-x}{3-3x} = e^t$

$\Rightarrow 3 - x = e^t(3 - 3x) \quad \Rightarrow \quad (3e^t - 1)x = 3e^t - 3 \quad \Rightarrow$

$$x = \frac{3(e^t - 1)}{3e^t - 1}$$

Or $\ln(3-x) - \ln(1-x) = t + \ln 3 \quad \Rightarrow \quad = \frac{3(e^t-1)}{3e^t-1}$

c.B1 $x = 0.906$ grams

i.e. Substitutes $t = 2$ leading to a value of x e.g $\quad x = \frac{3(e^2-1)}{3e^2-1} = 0.906$

d.B1 $x = 1$

e.g. as $t \to \infty \quad x = \frac{e^t(3-\frac{3}{e^t})}{e^t(3-\frac{1}{e^t})} = \frac{3-0}{3-0} = 1$

(Total for Question 14 is 9 marks)

15. The first three terms of a geometric series where θ is a constant are

$$-8\sin\theta, \quad 3-2\cos\theta \quad \text{and} \quad 4\cot\theta$$

a. Show that $4\cos^2\theta + 20\cos\theta + 9 = 0$

(3)

Given that θ lies in the interval $90^0 < \theta < 180^0$,

b. Find the value of θ.

(2)

c. Hence prove that this series is convergent.

(2)

d. Find S_∞, in the form $a(1-\sqrt{3})$

(2)

a.M1 For the key step in using the ratio of $\frac{a_2}{a_1} = \frac{a_3}{a_2}$

e.g. $\frac{3-2\cos\theta}{-8\sin\theta} = \frac{4\cot\theta}{3-2\cos\theta}$

dM1 Cross multiplies and uses $\cot\theta \times \sin\theta = \cos\theta$

e.g. $(3-2\cos\theta)^2 = -32\cot\theta\sin\theta \quad \Rightarrow \quad (3-2\cos\theta)^2 = -32\cos\theta$

A1 Proceeds to the given answer including the "0" with no errors and sufficient working shown.

e.g. $9 - 12\cos\theta + 4\cos^2\theta = -32\cos\theta \quad \Rightarrow 4\cos^2\theta + 20\cos\theta + 9 = 0$

b.M1 Attempts to solve $4\cos^2\theta + 20\cos\theta + 9 = 0$. Must be clear they have found $\cos\theta$ and not e.g. just t from $4t^2 + 20t + 9 = 0$. Working does not need to be seen but must solve a 3TQ correctly.

e.g. $4\cos^2\theta + 20\cos\theta + 9 = 0 \Rightarrow (2\cos\theta + 1)(2\cos\theta + 9) = 0$

$\Rightarrow \quad \cos\theta = -\frac{1}{2}, \cos\theta = -\frac{9}{2}$

A1 $\theta = 120^0$ and no other values.

c. B1 States that $a = -4\sqrt{3}$ or $r = -\frac{1}{\sqrt{3}}$ or $-\frac{\sqrt{3}}{3}$

e.g. $a = -8\sin 120^0 = -4\sqrt{3}$ and $r = \frac{3-2\cos 120^0}{-8\sin 120^0} = \frac{4}{-4\sqrt{3}} = -\frac{1}{\sqrt{3}}$

B1 States that $|r| < 1$

e.g. $-\frac{1}{\sqrt{3}}$ lies between -1 and 1

d. M1 Uses both values of "a" and "r" with the equation $S_\infty = \frac{a}{1-r} = \frac{-4\sqrt{3}}{1-(-\frac{1}{\sqrt{3}})}$ to create an

expression involving surds where a and r have come from appropriate work and $|r|$ <1

Rationalises denominator. The denominator must be of the form $a \pm b\sqrt{3}$ or

equivalent e.g. $a \pm \frac{b}{\sqrt{3}}$

Note that stating e.g. $\frac{p}{a+b\sqrt{3}} \times \frac{a-b\sqrt{3}}{a-b\sqrt{3}}$ or $\frac{p}{a+\frac{b}{\sqrt{3}}} \times \frac{a-\frac{b}{\sqrt{3}}}{a-\frac{b}{\sqrt{3}}}$ is sufficient.

e.g. $\frac{-4\sqrt{3}}{1+\frac{\sqrt{3}}{3}} \times \frac{1-\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}}$ or $\frac{-4\sqrt{3}}{1+\frac{1}{\sqrt{3}}} \times \frac{1-\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$

A1 Obtains $6(1-\sqrt{3})$

e.g. $\frac{-4\sqrt{3}}{1+\frac{\sqrt{3}}{3}} \times \frac{1-\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}} \Rightarrow \frac{-4\sqrt{3}+4}{1-\frac{3}{9}} \Rightarrow \frac{-4\sqrt{3}+4}{\frac{2}{3}} \Rightarrow \frac{3}{2}(4-4\sqrt{3})$

or $\frac{-4\sqrt{3}}{1+\frac{1}{\sqrt{3}}} \times \frac{1-\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} \Rightarrow \frac{-4\sqrt{3}+4}{1-\frac{1}{3}} \Rightarrow \frac{-4\sqrt{3}+4}{\frac{2}{3}} \Rightarrow \frac{3}{2}(4-4\sqrt{3})$

(Total for Question 15 is 9 marks)

16.

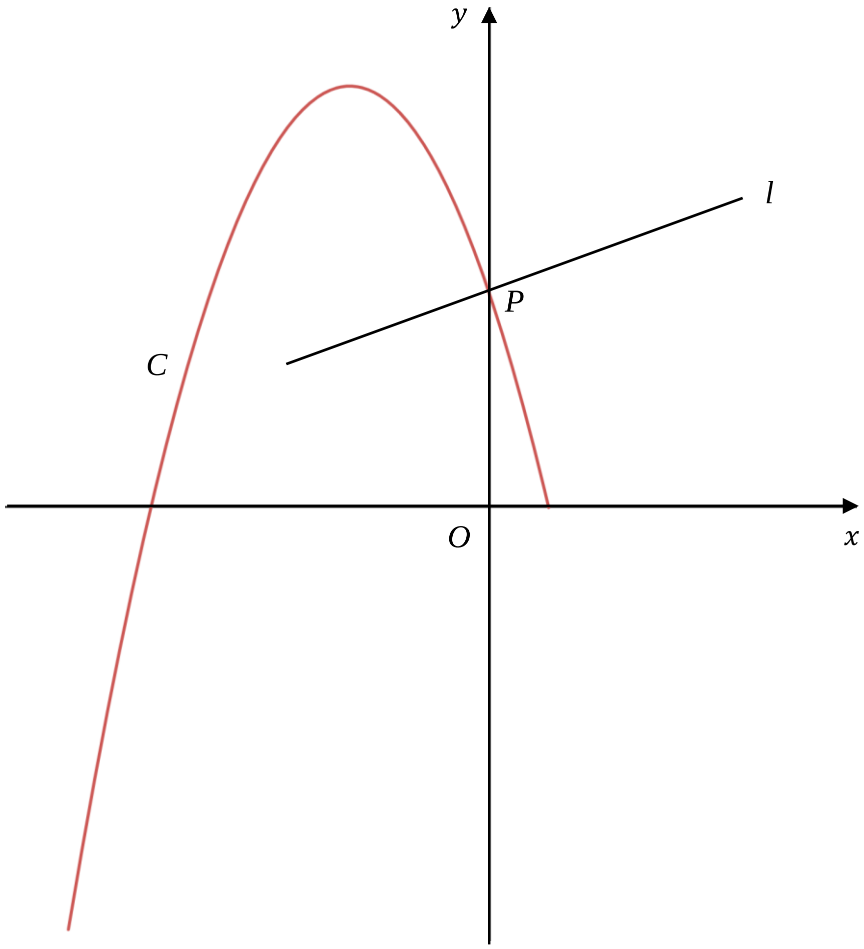


Figure 3

Figure 3 shows a sketch of the curve C with parametric equations

$x = -3 + 6\sin\theta$, $y = 9\cos 2\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{4}$

where θ is a parameter.

a. Find an expression for $\frac{dy}{dx}$ in terms of θ

(2)

The line l is normal to C at the point P where $\theta = \frac{\pi}{6}$

b. Show that an equation for l is

$$y = \frac{1}{3}x + \frac{9}{2}$$

(3)

c. The cartesian equation for the curve C can be written in the form

$$y = a - \frac{1}{2}(x + b)^2$$

where a and b are integers to be found.

(2)

The straight line with equation

$$y = \frac{1}{3}x + k$$

where k is a constant intersects C at two distinct points.

d. Find the range of possible values for k.

(5)

a.M1 For the key step of attempting $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. There must be some attempt to differentiate both parameters.

e.g. $\frac{dy}{d\theta} = p\sin 2\theta$ $\quad \frac{dx}{d\theta} = q\cos\theta$ where p, q are constants $\Rightarrow \frac{dy}{dx} = \frac{p\sin 2\theta}{q\cos\theta}$

A1 $\frac{dy}{dx} = \frac{-18\sin 2\theta}{6\cos\theta} = \frac{-3\sin 2\theta}{\cos\theta}$ Correct expression in any form is acceptable.

b.M1 For attempting to find the values of x and y and the gradient at $\theta = \frac{\pi}{6}$ and getting at least two correct. Follow through on their $\frac{dy}{dx}$ so allow any two of $x = 0, y = \frac{9}{2}$, $\frac{dy}{dx} = -3$ or their $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$

e.g. $x = -3 + 6\sin\theta = -3 + 6\sin\frac{\pi}{6} = 0$, $y = 9\cos 2\theta = 9\cos 2 \times \frac{\pi}{6} = \frac{9}{2}$

$$\frac{dy}{dx} = \frac{-3\sin 2\times\frac{\pi}{6}}{\cos\frac{\pi}{6}} = -3$$

M1 For a correct attempt at the normal equation using their x and y at $\theta = \frac{\pi}{6}$ with the negative reciprocal of their $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$ having made some attempt at $\frac{dy}{dx}$ and all correctly placed.

For attempts using $y = mx + c$ they must reach as far as a value for c using their x and y at $\theta = \frac{\pi}{6}$ with the negative reciprocal of their $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$ all correctly placed.

e.g. $y - \frac{9}{2} = \frac{1}{3}(x - 0)$ or $y = \frac{1}{3}x + c \Rightarrow \frac{9}{2} = \frac{1}{3}(0) + c \Rightarrow c = \frac{9}{2}$

A1 Proceeds with a clear argument to the given answer with no errors.

$$y = \frac{1}{3}x + \frac{9}{2}$$

c.M1 Attempts to use $\cos 2\theta = 1 - 2\sin^2\theta$ or equivalent to obtain an equation involving y and $(x+b)^2$

e.g. $y = 9(1 - 2\sin^2\theta) \Rightarrow y = 9\left(1 - 2\left(\frac{x+3}{6}\right)^2\right)$

A1 Obtains $y = 9 - \frac{1}{2}(x+3)^2$

e.g. $y = 9 - \frac{18}{36}(x+3)^2 \Rightarrow y = 9 - \frac{1}{2}(x+3)^2$

d.M1 A full attempt to find the **upper** limit for k.

$$9-\frac{1}{2}(x+3)^2=\frac{1}{3}x+k \quad \Rightarrow \quad 54-3(x+3)^2=2x+6k$$

$$\Rightarrow 3x^2+20x+6k-27=0 \quad \Rightarrow \quad b^2-4ac=20^2-4\times 3\times(6k-27)>0 \quad \Rightarrow k=\cdots$$

A1 $\quad k<\frac{181}{18}$

An **alternative method** using calculus for **upper** limit

M1 for $\frac{\mathrm{d}y}{\mathrm{d}x}=$ "a linear expression in x" sets $=\frac{1}{3}$, solves a linear equation to find x and then substitutes into the given result in (c) to find y and then uses $y=\frac{1}{3}x+k$ to find a value for k.

e.g. $y=9-\frac{1}{2}(x+3)^2 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=-(x+3), \quad -(x+3)=\frac{1}{3} \quad \Rightarrow \quad x=-\frac{10}{3}$

$$x=-\frac{10}{3} \quad \Rightarrow \quad y=9-\frac{1}{2}\left(-\frac{10}{3}+3\right)^2=\frac{161}{8}$$

$$y=\frac{1}{3}x+k \quad \Rightarrow \quad \frac{161}{8}=\frac{1}{3}\left(-\frac{10}{3}\right)+k \quad \Rightarrow \quad k=\cdots$$

A1 $\quad k<\frac{181}{18}$

An **alternative method** using parameters for **upper** limit

M1 Substituting parametric form of x and y into $y=\frac{1}{3}x+k$, uses $\cos 2\theta=1-2\sin^2\theta$ rearranges to3TQ form and attempts b^2-4ac ... 0 or e.g. $b^2-4ac>0$ or $b^2-4ac<0$ correctly to find a value for k

$$y=\frac{1}{3}x+k \quad \Rightarrow \quad 9\cos 2\theta=\frac{1}{3}(-3+6\sin\theta)+k$$

$$\Rightarrow \quad 9(1-2\sin^2\theta)=\frac{1}{3}(-3+6\sin\theta)+k \quad \Rightarrow \quad 18\sin^2\theta+2\sin\theta+k-10=0$$

$$\Rightarrow \quad b^2-4ac=2^2-4\times 18\times(k-10)=0 \quad \Rightarrow k=\frac{181}{18}$$

A1 $k=\frac{181}{18}$

M1 A full attempt to find the **lower** limit for k. This requires an attempt to find the value of x and the value of y using $\theta=\frac{\pi}{4}$, the substitution of these values into $y=\frac{1}{3}x+k$ and solves for k

e.g. $x=-3+6\sin\theta=-3+6\sin\frac{\pi}{4}=-3+3\sqrt{2}$, $y=9\cos 2\theta=9\cos 2\times\frac{\pi}{4}=0$

$$y=\frac{1}{3}x+k \quad \Rightarrow \quad 0=\frac{1}{3}\left(-3+3\sqrt{2}\right)+k \quad \Rightarrow \quad k=\cdots$$

A1 $k=1-\sqrt{2}$ Look for this value e.g. may appear in an equality.

A1 Deduces the correct range for k $\quad 1-\sqrt{2}\leq k<\frac{181}{18}$

Allow equivalent notation e.g. $\left(k<\frac{181}{18} \text{ and } k\geq 1-\sqrt{2}\right)$, $\left(k<\frac{181}{18}\cap k\geq 1-\sqrt{2}\right)$

and $\left[1-\sqrt{2},\frac{181}{18}\right)$

(Total for Question 16 is 12 marks)